

THE EQUIVARIANT RIEMANN-ROCH THEOREM AND THE GRADED TODD CLASS

MICHÈLE VERGNE

ABSTRACT. Let G be a torus with Lie algebra \mathfrak{g} and let M be a G -Hamiltonian manifold with Kostant line bundle \mathcal{L} and proper moment map. Let $\Lambda \subset \mathfrak{g}^*$ be the weight lattice of G . We consider a parameter $k \geq 1$ and the multiplicity $m(\lambda, k)$ of the quantized representation $RR_G(M, \mathcal{L}^k)$. Define $\langle \Theta(k), f \rangle = \sum_{\lambda \in \Lambda} m(\lambda, k) f(\lambda/k)$ for f a test function on \mathfrak{g}^* . We prove that the distribution $\Theta(k)$ has an asymptotic development $\langle \Theta(k), f \rangle \sim k^{\dim M/2} \sum_{n=0}^{\infty} k^{-n} \langle DH_n, f \rangle$ where the distributions DH_n are the twisted Duistermaat-Heckman distributions associated to the graded Todd class of M . When M is compact, and f polynomial, the asymptotic series is finite and exact.

1. INTRODUCTION

Let G be a torus with Lie algebra \mathfrak{g} . Identify \hat{G} to a lattice Λ of \mathfrak{g}^* . If $\lambda \in \Lambda$, we denote by g^λ the corresponding character of G . If $g = \exp(X)$ with $X \in \mathfrak{g}$, then $g^\lambda = e^{i\langle \lambda, X \rangle}$.

Let M be a prequantizable G -hamiltonian manifold with symplectic form Ω , Kostant line bundle \mathcal{L} , and moment map $\Phi : M \rightarrow \mathfrak{g}^*$. Assume M compact and of dimension $2d$. The Riemann-Roch quantization $RR_G(M, \mathcal{L})$ is a virtual finite dimensional representation of G , constructed as the index of a Dolbeaut-Dirac operator on M . The dimension of the space $RR_G(M, \mathcal{L})$ will be called the Riemann-Roch number of (M, \mathcal{L}) . The character of the representation of $RR_G(M, \mathcal{L})$ is a function on G , denoted by $RR_G(M, \mathcal{L})(g)$. We write

$$RR_G(M, \mathcal{L})(g) = \sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda) g^\lambda.$$

The typical example is the case where M is a projective manifold, and \mathcal{L} the corresponding ample bundle. Then

$$RR_G(M, \mathcal{L})(g) = \sum_{i=0}^d (-1)^i \text{Tr}_{H^i(M, \mathcal{O}(\mathcal{L}))}(g)$$

1

is the alternate sum of the traces of the action of g in the cohomology spaces of \mathcal{L} . In particular $\dim RR_G(M, \mathcal{L}) = \sum_{i=0}^d (-1)^i \dim H^i(M, \mathcal{O}(\mathcal{L}))$ is given by the Riemann-Roch formula.

It is natural to introduce the k^{th} power \mathcal{L}^k of the line bundle \mathcal{L} . Thus

$$RR_G(M, \mathcal{L}^k)(g) = \sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda, k) g^\lambda.$$

Assume $k \geq 1$. We associate to (M, \mathcal{L}) the distribution on \mathfrak{g}^* given by

$$\langle \Theta_M(k), f \rangle = \sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda, k) f(\lambda/k),$$

where f is a test function.

Example: When M is a toric manifold associated to the Delzant polytope P , then $\dim RR_G(M, \mathcal{L})$ is the number of integral points in the convex polytope P , and $\frac{1}{k^d} \langle \Theta_M(k), f \rangle$ is the Riemann sum of the values of f on the sample points $\frac{\Lambda}{k} \cap P$.

We prove that $\Theta_M(k)$ has an asymptotic behavior when the integer k tends to ∞ of the form

$$\Theta_M(k) \sim k^d \sum_{n=0}^{\infty} k^{-n} DH_n$$

where DH_n are distributions on \mathfrak{g}^* supported on $\Phi(M)$. We determine the distributions DH_n in terms of the decomposition of the equivariant Todd class $\text{Todd}(M)$ of M in its homogeneous components $\text{Todd}_n(M)$ in the graded equivariant cohomology ring of M . The distribution DH_0 is the Duistermaat-Heckmann measure. The asymptotics are exact when f is a polynomial. This generalizes the weighted Ehrhart polynomial for an integral polytope, and the asymptotic behavior of Riemann sums over convex integral polytopes established by Guillemin-Sternberg [8].

We then consider the case where M is a prequantizable G -hamiltonian manifold, not necessarily compact, but with proper moment map $\Phi : M \rightarrow \mathfrak{g}^*$. The formal quantization of (M, \mathcal{L}^k) ([20]) is defined by

$$RR_G(M, \mathcal{L}^k)(g) = \sum_{\lambda \in \Lambda} m_{\text{geo}}(\lambda, k) g^\lambda.$$

Here $m_{\text{geo}}(\lambda, k)$ is the geometric multiplicity function constructed by Guillemin-Sternberg in terms of the Riemann-Roch number of the reduced fiber $M_\lambda = \Phi^{-1}(\lambda)/G$ of the moment map. When M is compact,

Meinrenken-Sjamaar [10] proved that $m_{\text{rep}}(\lambda, k) = m_{\text{geo}}(\lambda, k)$, so this purely geometric definition extends the definition of $RR_G(M, \mathcal{L}^k)$ given in terms of index theory when M is compact.

Similarly, we construct distributions DH_n on \mathfrak{g}^* using the equivariant cohomology classes $\text{Todd}_n(M)$ and push-forwards by the proper map Φ . The main result of this announcement is that the distribution $\Theta_M(k)$ defined by

$$\langle \Theta_M(k), f \rangle = \sum_{\lambda \in \Lambda} m_{\text{geo}}(\lambda, k) f(\lambda/k),$$

is asymptotic to $k^d \sum_{n=0}^{\infty} k^{-n} DH_n$.

A similar result holds for Dirac operators twisted by powers of a line bundle \mathcal{L}^k .

Recall that we introduced a truncated Todd class (of the cotangent bundle T^*M) for determining the multiplicities of the equivariant index of any transversally elliptic operator on M [18]. Here the use of the parameter k allows us to have families of such equivariant indices, and the full series $\sum_{n=0}^{\infty} \text{Todd}_n(M)$ enters in the description of the asymptotic behavior. This is similar to the Euler-Mac Laurin formula evaluating sums of the values of a function at integral points of an interval involving all Bernoulli numbers. We finally give some information on the piecewise polynomial behavior of the distributions DH_n .

2. EQUIVARIANT COHOMOLOGY

Let N be a G -manifold and let $\mathcal{A}(N)$ be the space of differential forms on N , graded by its exterior degree. Following [3] and [19], an equivariant form is a G -invariant smooth function $\alpha : \mathfrak{g} \rightarrow \mathcal{A}(N)$, thus $\alpha(X)$ is a differential form on N depending differentiably of $X \in \mathfrak{g}$. Consider the operator

$$d_{\mathfrak{g}}\alpha(X) = d\alpha(X) - \iota(v_X)\alpha(X) \quad (2.1)$$

where $\iota(v_X)$ is the contraction by the vector field v_X generated by the action of $-X$ on N . Then $d_{\mathfrak{g}}$ is an odd operator with square 0, and the equivariant cohomology is defined to be the cohomology space of $d_{\mathfrak{g}}$. It is important to note that the dependance of α on X may be C^∞ . If the dependance of α in X is polynomial, we denote by $H_G^*(N)$ the corresponding \mathbb{Z} -graded algebra. By definition, the grading of $P(X) \otimes \mu$, P a homogeneous polynomial and μ a differential form on N , is the exterior degree of μ plus twice the polynomial degree in X .

The Hamiltonian structure on M determines the equivariant symplectic form $\Omega(X) = \langle \Phi, X \rangle + \Omega$.

Choose a G -invariant Riemannian metric on M . This provides the tangent bundle TM with the structure of a Hermitian vector bundle. Let $J(A) = \det_{\mathbb{C}^d} \frac{e^A - 1}{A}$, an invariant function of $A \in \text{End}(\mathbb{C}^d)$. Then, $J(0) = 1$. Consider $\frac{1}{J(A)}$ and its Taylor expansion at 0:

$$\frac{1}{J(A)} = \det_{\mathbb{C}^d} \left(\frac{A}{e^A - 1} \right) = \sum_{n=0}^{\infty} B_n(A).$$

Each function $B_n(A)$ is an invariant polynomial of degree n on $\text{End}(\mathbb{C}^d)$ and by the Chern Weil construction, B_n determines an equivariant characteristic class $\text{Todd}_n(M)(X)$ on M of homogeneous degree $2n$. Remark that $\text{Todd}_0(M) = 1$. We define the formal series of equivariant cohomology classes:

$$\text{Todd}(M)(X) = \sum_{n=0}^{\infty} \text{Todd}_n(M)(X).$$

For X small enough, the series is convergent, and $\text{Todd}(M)(X)$ is the equivariant Todd class of M . In particular $\text{Todd}(M)(0)$ is the usual Todd class of M .

In the rest of this note, using the Lebesgue measure $d\xi$ determined by the lattice Λ , we may identify distributions and generalized functions on \mathfrak{g}^* and we may write $\langle \theta, f \rangle = \int_{\mathfrak{g}^*} \theta(\xi) f(\xi) d\xi$ for the value of a distribution θ on a test function f on \mathfrak{g}^* .

3. THE COMPACT CASE

Let M be a compact G -Hamiltonian manifold. Recall (see [2]) the "delocalized Riemann-Roch formula". For $X \in \mathfrak{g}$ sufficiently small, we have

$$RR_G(M, \mathcal{L})(\exp X) = \frac{1}{(2i\pi)^d} \int_M e^{i\Omega(X)} \text{Todd}(M)(X).$$

Here $i = \sqrt{-1}$.

For each integer n , consider the analytic function on \mathfrak{g} given by

$$\theta_n(X) = \frac{1}{(2i\pi)^d} \int_M e^{i\Omega(X)} \text{Todd}_n(M)(X).$$

Thus when $X \in \mathfrak{g}$ is small, then $\sum_{n=0}^{\infty} \theta_n(X)$ is a convergent series with sum the equivariant Riemann-Roch index $RR_G(M, \mathcal{L})(\exp X)$. When $n = 0$,

$$\theta_0(X) = \frac{1}{(2i\pi)^d} \int_M e^{i\Omega(X)}$$

is the equivariant volume of M , and the Fourier transform DH_0 of θ_0 is the Duistermaat-Heckmann measure of M , a piecewise polynomial distribution on \mathfrak{g}^* .

Theorem 3.1. *Let DH_n be the Fourier transform of θ_n . Then DH_n is a distribution supported on $\Phi(M)$. For any polynomial function P of degree N on \mathfrak{g}^* , we have*

$$\sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda) P(\lambda) = \sum_{n \leq N+d} \int_{\mathfrak{g}^*} DH_n(\xi) P(\xi) d\xi.$$

In particular, we have the following Euler-MacLaurin formula for the Riemann-Roch number of (M, \mathcal{L}) :

$$\dim RR_G(M, \mathcal{L}) = \sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda) = \int_{\mathfrak{g}^*} \sum_{n \leq d} DH_n(\xi) d\xi.$$

We now give a theorem for smooth functions.

Theorem 3.2. *When the integral parameter k tends to ∞ , the distribution $\Theta_M(k)$ admits the asymptotic expansion*

$$\Theta_M(k) \sim k^d \sum_{n=0}^{\infty} k^{-n} DH_n.$$

Let us sketch the proof of Theorems 3.1 and 3.2. It is easy to see that the distributions DH_n are supported on the image $\Phi(M)$ of M by the moment map. Furthermore, it follows from the piecewise quasi-polynomial behavior of the function $m_{\text{rep}}(\lambda, k)$ that for P a homogeneous polynomial of degree N , the sum $\sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda, k) P(\lambda)$ is a quasi-polynomial function of $k \geq 1$ of degree less or equal than $N + d$. Thus Theorem 3.1 will be a consequence of Theorem 3.2 that we now prove.

The Fourier transform of $\Theta_M(k)$ is

$$\sum_{\lambda \in \Lambda} m_{\text{rep}}(\lambda, k) e^{i\langle \lambda, X/k \rangle} = RR_G(M, \mathcal{L}^k)(\exp(X/k)).$$

Against a test function ϕ of X , when k is large, this is

$$\begin{aligned} & \frac{1}{(2i\pi)^d} \int_{M \times \mathfrak{g}} e^{ik\Omega + ik\Phi(X/k)} \text{Todd}(M)(X/k) \phi(X) dX \\ &= \sum_{n,m} \frac{1}{(2i\pi)^d} \int_M e^{i\Phi(X)} \frac{1}{m!} k^m (i\Omega)^m \text{Todd}_n(M)(X/k) \phi(X) dX. \end{aligned}$$

For each m , only the term of differential degree $2d - 2m$ of $\text{Todd}_n(M)$ contributes to the integral, and this term is homogeneous in X of degree $n + m - d$. This implies the result.

Let us give an example of the asymptotic expansion.

Let $P_1(\mathbb{C})$ equipped with the torus action $g([z_1, z_2]) = [gz_1, z_2]$ of $g = e^{i\theta}$, in homogeneous coordinates. We consider $M = P_1(\mathbb{C}) \times P_1(\mathbb{C})$ with diagonal action, and let \mathcal{L} be its Kostant line bundle. Then we have

$$RR(M, \mathcal{L}^k)(g) = \sum_{j \in \mathbb{Z}} m_{\text{rep}}(j, k) g^j$$

with

$$m_{\text{rep}}(j, k) = \begin{cases} 0 & \text{if } j < -2k, \\ 2k + 1 + j & \text{if } -2k \leq j \leq 0, \\ 2k + 1 - j & \text{if } 0 \leq j \leq 2k, \\ 0 & \text{if } j > 2k. \end{cases}$$

We have

$$\Theta(k) \sim k^2(DH_0 + \frac{1}{k}DH_1 + \frac{1}{k^2}DH_2 + \frac{1}{k^3}DH_3 + \dots)$$

Let us give the explicit formulae for DH_0, DH_1, DH_2, DH_3 .

$$\langle DH_0, f \rangle = \int_{-2}^2 m(\xi) f(\xi) d\xi$$

with

$$m(\xi) = \begin{cases} 2 + \xi & \text{if } -2 \leq \xi \leq 0, \\ 2 - \xi & \text{if } 0 \leq \xi \leq 2, \end{cases}$$

$$\langle DH_1, f \rangle = \int_{-2}^2 f(\xi) d\xi,$$

$$\langle DH_2, f \rangle = \frac{5}{12}f(-2) + \frac{1}{6}f(0) + \frac{5}{12}f(2),$$

$$\langle DH_3, f \rangle = -\frac{1}{12}f'(-2) + \frac{1}{12}f'(2).$$

We now sketch another proof of Theorem 3.2 which can be extended to the non compact case. We use Paradan's decomposition ([15],[14], see also [17]) of $RR_G(M, \mathcal{L})$ in a sum of simpler characters supported on cones. Let us consider a generic value r of the moment map, and choose a scalar product on \mathfrak{g}^* . Then there exists a certain finite subset $\mathcal{B}(r)$ of \mathfrak{g}^* , and for each $\beta \in \mathcal{B}(r)$, a cone $C(\beta)$ in \mathfrak{g}^* and an (infinite dimensional) representation $P_{\beta,k}$ such that

$$RR_G(M, \mathcal{L}^k) = \sum_{\beta \in \mathcal{B}(r)} P_{\beta,k}.$$

Here $P_{\beta,k}(g) = \sum_{\lambda \in \Lambda \cap kC(\beta)} m_{\text{rep},\beta}(\lambda, k)g^\lambda$. Thus $\Theta(k)$ is decomposed in $\sum_{\beta \in \mathcal{B}(r)} \Theta_\beta(k)$. Similarly each distribution DH_n is decomposed as $DH_n = \sum_{\beta \in \mathcal{B}(r)} DH_{n,\beta}$ and the support of $DH_{n,\beta}$ is contained in the cone C_β . It is easily verified that, for each β , the distribution $\Theta_\beta(k)$ is asymptotic to $k^d \sum_{n=0}^{\infty} k^{-n} DH_{n,\beta}$. Here we use the explicit Euler MacLaurin expansion on half lines, and convolutions of such distributions. The proof is entirely similar to the case of a polytope given in [4].

Let us return to the example of the case of $M = P_1(\mathbb{C}) \times P_1(\mathbb{C})$, for $r < 0$ a small negative number. Then $\mathcal{B}(r) = \{-2, r, 0, 2\}$. We have

$$P_{\beta=-2,k}(g) = - \sum_{j < -2k} (2k + 1 + j)g^j,$$

$$P_{\beta=r,k}(g) = \sum_{j=-\infty}^{j=\infty} (2k + 1 + j)g^j,$$

$$P_{\beta=0,k}(g) = -2 \sum_{j>0} jg^j,$$

$$P_{\beta=2,k}(g) = \sum_{j>2k} (j - (2k + 1))g^j.$$

Consider, for example, the asymptotic development of the distribution

$$\langle \Theta_{\beta=2}(k), f \rangle = \sum_{j>2k} (j - (2k + 1))f(j/k).$$

It is easy to see that this distribution is the convolution $K(k) * K(k)$ where $K(k)$ is the distribution defined by $\langle K(k), f \rangle = \sum_{j>k} f(j/k) = \sum_{j \geq k} f(j/k) - f(1)$. We then use the explicit exact Euler Mac-Laurin formula to evaluate the distribution $K(k)$, thus its convolution. In particular the Fourier transform of $K(k) * K(k)$ coincides with the analytic function $\frac{e^{2ix}}{(1-e^{-ix/k})^2}$ for $(1 - e^{-ix/k}) \neq 0$. As is natural, the asymptotic series of distributions $q^{-d} \sum_{n=0}^{\infty} q^n DH_{\beta=2,n}$ is the unique series of distributions supported on $\xi \geq 2$ and with Fourier transform, for $x \neq 0$, the Laurent series in q of $\frac{e^{2ix}}{(1-e^{-iqx})^2}$ at $q = 0$.

4. PROPER MOMENT MAPS

Consider the case where M is non necessarily compact, but $\Phi : M \rightarrow \mathfrak{g}^*$ is a proper map. One can then define ([20],[13]) the formal geometric quantification of M with respect to the line bundle \mathcal{L}^k to be

$$RR_{G,\text{geo}}(M, \mathcal{L}^k)(g) = \sum_{\lambda \in \Lambda} m_{\text{geo}}(\lambda, k)g^\lambda,$$

using a function $m_{\text{geo}}(\xi)$ on \mathfrak{g}^* . The definition of the function $m_{\text{geo}}(\xi)$ is due to Guillemin-Sternberg [7]. Let us recall its delicate definition ([10], see also [16]). There is a closed set \mathcal{A} , union of affine hyperplanes, such that if r is in the complement of \mathcal{A} , then either r is not in $\Phi(M)$ or r is a regular value of Φ . Consider the open subset $\mathfrak{g}_{\text{reg}}^* = \mathfrak{g}^* \setminus \mathcal{A}$. When $\xi \in \mathfrak{g}_{\text{reg}}^*$ but not in $\Phi(M)$, $m_{\text{geo}}(\xi)$ is defined to be 0. If $\xi \in \mathfrak{g}_{\text{reg}}^* \cap \Phi(M)$, the reduced fiber $M_\xi = \Phi^{-1}(\xi)/G$ is a compact symplectic orbifold, and $m_{\text{geo}}(\xi)$ is defined to be a sum of integrals on the various strata of the compact orbifold M_ξ . When $\lambda \in \mathfrak{g}_{\text{reg}}^* \cap \Lambda \cap \Phi(M)$, then M_λ is a prequantizable compact symplectic orbifold and $m_{\text{geo}}(\lambda)$ is the Riemann-Roch number of M_λ equipped with its Kostant orbifold line bundle. Let $\lambda \in \Lambda$ be any point in $\Phi(M)$. Choose a vector ϵ , such that $\lambda + t\epsilon$ is in $\Phi(M) \cap \mathfrak{g}_{\text{reg}}^*$ for any $t > 0$ and sufficiently small. It can be proved, using the wall crossing formulae of Paradan [12], that $(\lim_{\epsilon} m_{\text{geo}})(\lambda) = \lim_{t \rightarrow 0, t \rightarrow 0} m_{\text{geo}}(\lambda + t\epsilon)$ is independent of the choice of such an ϵ . This allows us to define $m_{\text{geo}}(\lambda)$ by "continuity on $\Phi(M)$ " for any $\lambda \in \Lambda$.

The $[Q, R] = 0$ theorem ([10], [9], [11]) asserts that $RR_{G, \text{geo}}(M, \mathcal{L})$ coincides with a representation of G defined using index theory. In particular $RR_{G, \text{geo}}(M, \mathcal{L})$ coincides with $RR_G(M, \mathcal{L})$ when M is compact. However in the rest of this note, we only use the geometric definition of $RR_{G, \text{geo}}(M, \mathcal{L})$.

Replacing \mathcal{L} by \mathcal{L}^k , and the moment map Φ by $k\Phi$, define the distribution, with parameter k ,

$$\langle \Theta_M(k), f \rangle = \sum_{\lambda \in \Lambda} m_{\text{geo}}(\lambda, k) f(\lambda/k).$$

As in the compact case, the asymptotic behavior of $\Theta_M(k)$ is determined by the graded Todd class, using push-forwards by the proper map Φ . Indeed if α is an equivariant cohomology class with polynomial coefficients, then the Duistermaat-Heckman twisted distribution $DH(M, \Phi, \alpha)$ is well defined by the formula

$$\langle DH(M, \Phi, \alpha), f \rangle = \frac{1}{(2i\pi)^d} \int_{M \times \mathfrak{g}} e^{i\Omega(X)} \alpha(X) \hat{f}(X) dX$$

where $\hat{f}(X) = \int_{\mathfrak{g}^*} e^{i\langle \xi, X \rangle} f(\xi) d\xi$ is the Fourier transform of the test function $f(\xi)$ (see [6]). It is a distribution supported on $\Phi(M)$.

Definition 4.1. We define DH_n to be the distribution on \mathfrak{g}^* associated to the equivariant cohomology class $\text{Todd}_n(M)$:

$$\langle DH_n, f \rangle = \frac{1}{(2i\pi)^d} \int_{M \times \mathfrak{g}} e^{i\Omega(X)} \text{Todd}_n(X) \hat{f}(X) dX.$$

The distribution DH_0 is the Duistermaat-Heckman measure, a locally polynomial function.

The distribution DH_n is given by a polynomial function on each connected component of the open set $\mathfrak{g}_{\text{reg}}^*$. Its restriction to $\mathfrak{g}_{\text{reg}}^*$ vanishes when $n > d - \dim G$. Furthermore, if all stabilizers of points of M are connected, it follows from Witten non abelian localization theorem that

$$m_{\text{geo}}(\lambda, k) = k^d \sum_{n=0}^{\infty} k^{-n} DH_n(\lambda/k)$$

when λ/k is a regular value of Φ . Otherwise, it can be defined by limit of the function $m_{\text{geo}}(\xi, k) = k^d \sum_{n=0}^{\infty} k^{-n} DH_n(\xi/k)$, along $\xi = \lambda + t\epsilon_\lambda$ and $t > 0, t \rightarrow 0$, where the direction ϵ_λ is chosen to be arbitrary if λ does not belong to $\Phi(M)$, or in such a way that $\lambda + t\epsilon_\lambda$ stays in $\Phi(M)$ if $\lambda \in \Phi(M)$. Similar formulae can be given without assumption on connected stabilizers.

We can see that, for any n , the distributions DH_n can be expressed (but not uniquely) as derivatives of locally polynomial functions associated to symplectic submanifolds M^T where T are subtori of G .

The main result of this note is the following theorem.

Theorem 4.2. *When the integer k tends to ∞ ,*

$$\Theta_M(k) \sim k^d \sum_{n=0}^{\infty} k^{-n} DH_n.$$

Let us sketch the proof of this theorem, in the case where each stabilizer is connected. We use Paradan decomposition formula [14], [15]. We choose r a generic element of $\mathfrak{g}_{\text{reg}}^*$. As in the compact case, there is a locally finite set $\mathcal{B}(r) \subset \Phi(M)$, cones C_β , and decompositions

$$DH_n = \sum_{\beta \in \mathcal{B}(r)} DH_{n,\beta}$$

where $DH_{n,\beta}$ are supported on C_β . The functions $DH_{n,\beta}$ are given by polynomial functions on each connected component of $\mathfrak{g}_{\text{reg}}^*$ and vanishes on $\mathfrak{g}_{\text{reg}}^*$ when $n > d - \dim G$. Thus the locally polynomial function $A_\beta(\xi, k) = k^d \sum_{n=0}^{\infty} k^{-n} DH_{n,\beta}(\xi/k)$ is well defined when $\xi/k \in \mathfrak{g}_{\text{reg}}^*$. For each $\beta \in \mathcal{B}(r)$, choose a direction ϵ_β such that $\beta + t\epsilon_\beta$ is in $\Phi(M) \cap \mathfrak{g}_{\text{reg}}^*$ for $t > 0$ small. Then $w_\beta(\lambda, k) = \lim_{t>0, t \rightarrow 0} A_\beta(\lambda + t\epsilon_\beta, k)$ is well defined. Define

$$P_{\beta,k}(g) = \sum_{\lambda \in \Lambda} w_\beta(\lambda, k) g^\lambda$$

and

$$\langle \Theta_{\beta, \text{geo}}(k), f \rangle = \sum_{\lambda \in \Lambda} w_{\beta}(\lambda, k) f(\lambda/k).$$

As before, it is easy to see that $\Theta_{\beta, \text{geo}}(k) \sim k^d \sum_{n=0}^{\infty} k^{-n} DH_{n, \beta}$. Here we use the following "continuity" result on partition function (see for example [5]). Let Δ be a unimodular list of non zero vectors in Λ , and $\gamma \in \mathfrak{g}$ generic. There is a unique function K (the Kostant partition function) on Λ supported on the half space $\langle \xi, \gamma \rangle \geq 0$ and such that $\sum_{\lambda \in \Lambda} K(\lambda) g^{\lambda} = \prod_{\alpha \in \Delta} \frac{1}{1-g^{\alpha}}$ for g in the open set $\prod_{\alpha \in \Delta} (1-g^{\alpha}) \neq 0$. Let $d = |\Delta|$. Consider the Laurent series expansion in q

$$\prod_{\alpha \in \Delta} \frac{1}{1 - e^{q\langle \alpha, X \rangle}} = q^{-d} \sum_{n=0}^{\infty} q^n U_n(X)$$

and the distributions D_n on \mathfrak{g}^* supported on the half space $\langle \xi, \gamma \rangle \geq 0$, such that

$$\int_{\mathfrak{g}^*} D_n(\xi) e^{i\langle \xi, X \rangle} = U_n(X)$$

when $\prod_{\alpha} \langle \alpha, X \rangle \neq 0$. Define $T(\xi) = \sum_{n=0}^{\infty} D_n(\xi)$, which is well defined outside a system of hyperplanes. Then for any $\lambda \in \Lambda$, and ϵ_{Δ} generic and belonging to the cone $\text{Cone}(\Delta)$ generated by Δ , we have $K(\lambda) = \lim_{t>0, t \rightarrow 0} T(\lambda + t\epsilon_{\Delta})$.

Define $P_{r,k} = \sum_{\beta \in \mathcal{B}(r)} P_{\beta,k}$. It remains to see that $P_{r,k} = RR_{G, \text{geo}}(M, \mathcal{L}^k)$. This is not immediate, since we do not have a global representation theoretic object for describing $RR_{G, \text{geo}}(M, \mathcal{L}^k)$. Each coefficient $m_{\text{geo}}(\lambda, k)$ is defined using a limit direction depending on λ while each $w_{\beta}(\lambda, k)$ is defined using the same limit direction (depending on β) for any λ . So additivity is not clear. However, we can prove that $P_{r,k}$ is independent of r , using [12]. This is very similar to the technique used in [1] to establish decompositions à la Paradan of characteristic functions of polyhedra. It then follows that $P_{r,k} = RR_{G, \text{geo}}(M, \mathcal{L}^k)$. Indeed for each connected component \mathfrak{c} of $\mathfrak{g}_{\text{reg}}^*$ contained in $\Phi(M)$, we choose r in \mathfrak{c} . In the decomposition $P_{r,k} = \sum_{\beta \in \mathcal{B}(r)} P_{\beta,k}$, the term $w_{\beta}(\lambda, k)$ for $\beta = r \in \mathcal{B}(r)$ is the polynomial function coinciding with $m_{\text{geo}}(\lambda, k)$ for $\lambda \in k\bar{\mathfrak{c}}$. The other terms $w_{\beta}(\lambda, k)$ for $\beta \in \mathcal{B}(r)$ and $\beta \neq r$ vanishes when $\lambda \in k\bar{\mathfrak{c}}$ ([14], see also [17]).

A quicker route, but less instructive, for determining asymptotics of $\Theta_{M, \text{geo}}$ would be to take a test function with small support around a point $r \in \mathfrak{g}^*$. Then we can choose ϵ_{λ} coinciding with ϵ_{β} for all $\beta \in \mathcal{B}(r)$ and in the support of the test function f . The additivity is immediate on those β .

REFERENCES

- [1] José Agapito and Leonor Godinho, *New polytope decompositions and Euler-Maclaurin formulas for simple integral polytopes*, Adv. Math. **214** (2007), no. 1, 379–416, DOI 10.1016/j.aim.2007.02.008.
- [2] Nicole Berline, Ezra Getzler, and Michèle Vergne, *Heat kernels and Dirac operators*, Grundlehren Text Editions, Springer-Verlag, Berlin, 2004. Corrected reprint of the 1992 original.
- [3] Nicole Berline and Michèle Vergne, *Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante*, C. R. Acad. Sci. Paris Sér. I Math. **295** (1982), no. 9, 539–541 (French, with English summary).
- [4] Nicole Berline and Michèle Vergne, *Local asymptotic Euler-Maclaurin expansion for Riemann sums over a semi-rational polyhedron*, arXiv **1502.01671** (2015).
- [5] C. De Concini, C. Procesi, and M. Vergne, *Box splines and the equivariant index theorem*, J. Inst. Math. Jussieu **12** (2013), no. 3, 503–544, DOI 10.1017/S1474748012000734.
- [6] ———, *The infinitesimal index*, J. Inst. Math. Jussieu **12** (2013), no. 2, 297–334, DOI 10.1017/S1474748012000722.
- [7] V. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. **67** (1982), no. 3, 515–538, DOI 10.1007/BF01398934.
- [8] Victor Guillemin and Shlomo Sternberg, *Riemann sums over polytopes*, Ann. Inst. Fourier (Grenoble) **57** (2007), no. 7, 2183–2195 (English, with English and French summaries), Festival Yves Colin de Verdière.
- [9] Xiaonan Ma and Weiping Zhang, *Geometric quantization for proper moment maps: the Vergne conjecture*, Acta Math. **212** (2014), no. 1, 11–57, DOI 10.1007/s11511-014-0108-3.
- [10] Eckhard Meinrenken and Reyer Sjamaar, *Singular reduction and quantization*, Topology **38** (1999), no. 4, 699–762, DOI 10.1016/S0040-9383(98)00012-3.
- [11] Paul-Émile Paradan, *Formal geometric quantization II*, Pacific J. Math. **253** (2011), no. 1, 169–211, DOI 10.2140/pjm.2011.253.169.
- [12] Paul-Émile Paradan, *Wall-crossing formulas in Hamiltonian geometry*, Geometric aspects of analysis and mechanics, Progr. Math., vol. 292, Birkhäuser/Springer, New York, 2011, pp. 295–343.
- [13] Paul-Émile Paradan, *Formal geometric quantization*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 1, 199–238 (English, with English and French summaries).
- [14] Paul-Émile Paradan, *Localization of the Riemann-Roch character*, J. Funct. Anal. **187** (2001), no. 2, 442–509, DOI 10.1006/jfan.2001.3825.
- [15] ———, *Formules de localisation en cohomologie équivariante*, Compositio Math. **117** (1999), no. 3, 243–293, DOI 10.1023/A:1000602914188 (French, with English and French summaries).
- [16] Paul-Émile Paradan and Michèle Vergne, *Witten non abelian localization for equivariant K -theory, and the $[Q, R]=0$ theorem*, arXiv **1504.07502** (2015).
- [17] Michèle Vergne and Andras Szenes, *$[Q, R] = 0$ and Kostant partition functions*, arXiv **1006.4149** (2010).
- [18] Michèle Vergne, *Formal equivariant \hat{A} class, splines and multiplicities of the index of transversally elliptic operators*, Izvestiya Mathematics **80** (2016).

- [19] Edward Witten, *Supersymmetry and Morse theory*, J. Differential Geom. **17** (1982), no. 4, 661–692 (1983).
- [20] Jonathan Weitsman, *Non-abelian symplectic cuts and the geometric quantization of noncompact manifolds*, Lett. Math. Phys. **56** (2001), no. 1, 31–40, DOI 10.1023/A:1010907708197. EuroConférence Moshé Flato 2000, Part I (Dijon). MR1848164

UNIVERSITÉ DENIS-DIDEROT-PARIS 7, INSTITUT DE MATHÉMATIQUES DE JUSSIEU,
C.P. 7012, 2 PLACE JUSSIEU, F-75251 PARIS CEDEX 05
E-mail address: `michele.vergne@imj-prg.fr`